MATHSEON
COME, CRASH & CONQUER

BILINEAR TRANSFORMATIONS

Conformal Mapping

A function $w = f(z)$ is said to be conformal at z_0 if curve -in the z - plane passing through z_0& image curve in w-plane passing $f(z_0)$ preserves the angle in magnitude & sense of rotation (orientation) of angle.

Theorem

Let $f \in \boldsymbol{H}(D)$ i.e, analytic on D & $z_0 \in D$ such that $f'(z_0) \neq 0$. Then f is conformal at z_0.

Example

$f(z) = e^z$ is conformal everywhere since e^z is an entire function & $f'(z) = e^z \neq 0 \; \forall z \in \mathbb{C}$.

Critical Point

If f is non constant analytic at z_0&$f'(z_0) = 0$, then the conformal character fails at z_0, such a point z_0 is called a critical point of f.

Example

Consider $w = f(z) = \sin\ z$ since $f(z)$ is entire, so for every $z_0 \in \mathbb{C}$ is a regular point $\Rightarrow z = k\pi : k \in \mathbb{Z}$ are critical points of $f(z)$.

Bilinear/ Linear Fractional/ Mobius Transformation

The map

$$w = \frac{az+b}{cz+d},\ ad - bc \neq 0 \qquad (i)$$

Where $a, b, c\&d$ are complex constants, is called a linear fractional transformation, or Mobius transformation & equation (i) can also be written as

$$Azw + bz + cz + D = 0,\ (AD - BC \neq 0)$$

and vice versa.

Note

A mobius transformation is simply a composition of one, some or all of the following special types of transformation.

- **Translation**: It is a map of the form $z \mapsto z + \alpha, \alpha \in \mathbb{C} \setminus \{0\}$. If $\alpha = 0$ then it is an identity.

- **Magnification or Contraction**: It is a map of the form $z \mapsto rz, r \in R-\{0\}$. For $r=1$, this is the identity map & for $r=0$ it is a constant map.
  Case (i) When $r>1$, then this is a "magnification".
  Case (ii) When $r<1$, then this is a contraction map.

Note

If $r<0$ then $w=rz$ gives the reflection through the origin followed by such a "magnification" or shrinking/contraction depending on $r<-1$ or $-1<r<0$.

- **Rotation**: It is a map of the form $z \mapsto e^{i\theta}\hat{z}; \theta \in \mathbb{R}$. This map produces a rotation through an angle about the origin with positive sense, $\theta>0$.

Note

The rotation coupled with magnification is referred to as Dilation: $z \mapsto az(a \neq 0)$.

- **Inversion**: It is a map of the form $z \mapsto \frac{1}{z}$ which produces a geometric inversion (or reciprocal map or the inversion map.)

Remark

❖ If we let $T(z)=T_{abcd}(z)$& if $\alpha \in \mathbb{C} \setminus \{0\}$, then $\alpha a, \alpha b, \alpha c, \alpha d$ correspond to the same mobius transformation as
$T_{abcd}(z)=T_{(a\alpha)(b\alpha)(c\alpha)(d\alpha)}(z)$
i.e., behavior of T does not change when a, b, c, d are multiplied by a non-zero constant.

❖ The mobius transformation $T(z)$ is analytic on $C \setminus \{d/c\}$.

❖ If $c=0$ then $T(z)=\frac{az+b}{cz+d}, ad-bc \neq 0$ reduces to $T(z)=\frac{a}{d}z+\frac{b}{d}=\alpha z+\beta(ad \neq 0, \alpha \neq 0)$& called a linear map.

❖ Every mobius transformation $T(z)$,
$T(z)=\frac{az+b}{c+d}, ad-bc \neq 0$, can be decompose as

$$T(z)=\left[a\left(z+\frac{d}{c}\right)+b-\frac{ad}{c}\right]\frac{1}{c\left(z+\frac{d}{c}\right)}$$

$$=\frac{a}{c}-\left(\frac{ad-bc}{c^2}\right)\frac{1}{\left(z+\frac{d}{c}\right)}, c \neq 0$$

❖ If $ad-bc=0$ then $T(z)$ is a constant map

Fixed Point

Let D be a subset of $\mathbb{C}_\infty$ and $f: D \to \mathbb{C}_\infty$. A point $z_0 \in D$ is said to be a fixed point of f if $f(z_0)=z_0$. The set of all fixed points of f is denoted by **Fix** $(\boldsymbol{f})$

Examples

- The function $\boldsymbol{f(z) = z^2}$ has exactly three fixed points, namely, 0,1 and ∞ whereas the function $f(z) = z^{-1}$ has two fixed points namely 1 and -1.

- The function $\boldsymbol{f(z) = z - 1}$ has no fixed points in $\mathbb{C}$ whereas it has one fixed point in $\mathbb{C}_\infty$, namely the point at ∞

- The function $\boldsymbol{f(z) = \frac{iz}{|z|}, z \neq 0}$, has no fixed points in $\mathbb{C} \setminus \{0\}$

Note

❖ Every non-constant real-valued continuous function $f: (-1,1) \to (-1,1)$ has a fixed point in $(-1,1)$. – However, a similar result does not hold for functions $f: \Delta: \Delta$. For example $\phi_\alpha: \Delta \to \Delta, |\alpha| = 1$, defined by

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}$$

has no fixed points in Δ.

Proposition

Every Mobius transformation $T: \mathbb{C}_\infty \to \mathbb{C}_\infty$ has at most two fixed points in $\mathbb{C}_\infty$ unless $T(z) \equiv z$. Equivalently, if a Mobius transformation leaves three points in $\mathbb{C}_\infty$ fixed, then it is none other than the identity function.

Corollary

If S and T are two Mobius transformations which agree at three distinct points of $\mathbb{C}_\infty$, then $S = T$.

Results

❖ Every mobius transformation maps circles and straight lines into circles and straight lines.

❖ Every Mobius transformation maps circles in $\mathbb{C}_\infty$ onto circles in $\mathbb{C}_\infty$.

❖ Under translation, magnification (scaling) & rotation, circles maps to circles & lines to lines.

❖ Under the function $w = \frac{1}{z}$, we have

- The image of a line through the origin is a line through the origin.
- The image of a line not through the origin is a circle through the origin.
- The image of a circle through the origin is a line not through the origin.
- The image of a circle not through the origin is a circle not through the origin.

Classification of Bilinear Transformation-on the basis of Normal Form

Let $w = T(z) = \frac{az+b}{cz+d}$ be a bilinear transformation.

Parabolic: The bilinear transformation with one fixed point is called parabolic i.e., $(a-d)^2 + 4bc = 0$.

Elliptic: A bilinear transformation with two fixed points i.e., $(a-d)^2 + 4bc \neq 0$ such that $|k| = 1, k \neq 1$ is said to be elliptic i.e., in the normal form k is of the form $k = e^{i\alpha}, \alpha \neq 0$.

Hyperbolic: A bilinear transformation with two fixed points i.e., $(a-d)^2 + 4bc \neq 0 \& k > 0, k \in \mathbb{R}$ is termed as hyperbolic.

Loxodromic: A bilinear transformation that is neither hyperbolic, elliptic, nor parabolic is called loxodromic, i.e., it has two fixed points & satisfies the condition $k = \alpha e^{i\alpha}, \alpha \neq 0, \alpha \neq 1$.

Cross Ratio

For the set of three distinct points z_1, z_2, z_3 of $\mathbb{C}_\infty$, the expression $\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-z_1)/(z-z_3)}{(z_2-z_1)/(z_2-z_3)}$ is called the cross-ratio of the four points $z, z_1, z_2, z_3 \&$ is denoted by (z, z_1, z_2, z_3).

Symmetric Point/Inverse Point

Let L be a line in $\mathbb{C}$. Two point $a \& a^*$ in $\mathbb{C}$ are said to be symmetric with respect to L if L is the perpendicular bisector of $[a, a^*]$ the line segment connecting $a \& a^*$.

Examples:

(i) Two points $z \& z^*$ are symmetric w.r.t. the real axis when $z^* = \bar{z}$.

(ii) Two points $z \& z^*$ are symmetric w.r.t. the imaginary axis iff $z^* = -\bar{z}$

Suppose that k is a circle $|z - z_0| = r$ in $\mathbb{C}$. Two points $a \& a^*$ are said to be symmetric w.r.t. circle k (or inverse points w.r.t. the circle k) iff $|a - z_0||a^* - z_0| = r^2 \& \text{Arg}\,(a - z_0) = \text{Arg}\,(a^* - z_0)$. i.e., $a \& a^*$ on the same ray emanating from the centre z_0 of k, & the product of their distances from the centre of the circle k is equal to the square of the radius of the circle.